TAUBERIAN THEOREMS

BY

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ABSTRACT

Tauberian constants and estimates are calculated for the difference of two linear transforms from the form (1.1) of the same function satisfying Tauberian conditions.

Applications for number-sequences, and connections with previous results are shown.

1. **Introduction.** Denote by $T(y)$ $(y \geqq 0)$ some integral-transform of a function $f(x)$ $(-\infty < x < +\infty)$ of the form

$$T(y) \equiv T_\beta(y) = \int_{-\infty}^{+\infty} f(x)d(1-\beta(y-x)) \tag{1.1}$$

where $\beta(x)$ is a function of bounded variation on $-\infty < x < +\infty$. In addition to Tauberian theorems, which give information about $\lim_x f(x)$ if $\lim_y T(y)$ exists, it is possible to find estimates concerning $| T_\beta(y) - f(x) |$ even if neither the existence of $\lim_y T(y)$ nor that of $\lim_x f(x)$ is assumed. Studies of such problems concerning number-sequences and the usual Abel-transform originated with the paper of Hadwiger [5], concerning integral-transforms with Agnew [2], Delange [3], Rajagopal [10], Jakimowski [8]. In Section II of the present paper we shall obtain a simple proof of estimates of $| T_\beta(y) - T_\gamma(\eta) |$ where T_β and T_γ are transforms satisfying certain general conditions, y and η tend to $+\infty$ with a connection on $y - \eta$. Our result contains as special cases many known results.

2. **Tauberian Constants.** The main result is the following:

THEOREM 1. *Let $f(x)$ $(-\infty < x < +\infty)$ be a real or complex-valued, continuous, almost everywhere differentiable function satisfying*

$$\lim_{x \to -\infty} f(x) = 0 \tag{2.1}$$

$$f'(x) = O(1) \text{ for } -\infty < x < +\infty \tag{2.2}$$

$$\int_{-\infty}^{x} f'(t)dt = f(x) \tag{2.3}$$

$$\overline{\lim_{x \to +\infty}} |f'(x)| = L < +\infty. \tag{2.4}$$

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Let $\beta(x)$ and $\gamma(x)$ be real functions of bounded variation over $-\infty < x < +\infty$, satisfying

$$(2.5)\qquad \lim_{x\to-\infty}\beta(x)=\lim_{x\to-\infty}\gamma(x)=0, \qquad \lim_{x\to+\infty}\beta(x)=\lim_{x\to+\infty}\gamma(x)=1$$

$$(2.6)\qquad \begin{cases}\beta(x)\in L_1(-\infty,0) & ; \quad \gamma(x)\in L_1(-\infty,0)\\ (1-\beta(x))\in L_1(0,+\infty); & \quad (1-\gamma(x))\in L_1(0,+\infty)\end{cases}$$

$$(2.7)\qquad \int_{-\infty}^{0}\int_{-\infty}^{u}|d\beta(x)|\,du < +\infty, \quad \int_{-\infty}^{0}\int_{-\infty}^{u}|d\gamma(x)|\,du < +\infty$$

and let $y=y(t)$ and $\eta=\eta(t)$ be positive increasing function with $\lim_{t\to\infty}y(t)=+\infty$, $\lim_{t\to\infty}\eta(t)=+\infty$ and

$$(2.8)\qquad \lim_{t\to+\infty}(y(t)-\eta(t))=q, \qquad -\infty<q<+\infty$$

then

$$(2.9)\qquad \overline{\lim_{t\to+\infty}}\,|T_\beta(y)-T_\gamma(\eta)|\leqq L\cdot A_q$$

where

$$(2.10)\qquad A_q=\int_{-\infty}^{+\infty}|\beta(x)-\gamma(x-q)|\,dx,$$

and A_q is the best possible constant satisfying (2.9), since there exists a real function $f(x)$, with $0<L<+\infty$ and such, that in (2.9) the equality sign holds.

Before giving the proof we mention some particular examples.

EXAMPLE (i). Let $\{s_k\}$ $(k\geqq 1)$ $(s_k=a_1+a_2+\ldots+a_k)$ be a real or complex sequence satisfying $ka_k=O(1)$, and define

$$(2.11)\qquad \beta(x)=e^{-e^{-x}},\quad \gamma(x)=\begin{cases}0 & x<0\\ 1 & x\geqq 0\end{cases}$$

$$(2.12)\qquad f(x)=\sum_{k=1}^{n}a_k+a_{n+1}\frac{x-\log n}{\log(n+1)-\log n},\quad \log n\leqq x<\log(n+1)$$

If we denote $e^{\eta}=\tau$, $e^{-e^{-y}}=\rho$, then it is easy to see that $T_\gamma(\eta)=s_{[\tau]}+o(1)$ and $T_\beta(y)=(1-\rho)\sum_{k=1}^{\infty}s_k\rho^k+o(1)$.

Thus we have by Theorem 1

$$\lim_{t\to+\infty}|s_{[\tau]}-A(\rho)|\leqq L\left(\int_{-\infty}^{-\log q}e^{-e^{-x}}dx+\int_{-\log q}^{+\infty}(1-e^{-e^{-x}})dx\right)$$

where $L = \overline{\lim}_{k\to\infty} |ka_k|$, $A(\rho)$ is the usual Abel-transform of the series $\{s_k\}$, $\tau \to +\infty$, $\rho \to 1$, $\tau(1-\rho) \to q$ as $t \to +\infty$. This is an equivalent form of a result of Agnew [7].

EXAMPLE. (ii). Let $\{s_k\}, f(x), \gamma(x)$ and L defined as in example (i).

Let for $\alpha > 0$,

$$\beta(x) = \begin{cases} 0 & x < 0 \\ (1-e^{-x})^{\alpha} & x \geqq 0 \end{cases}; \tag{2.13}$$

If we denote $e^y = u$, $e^\eta = \tau$, and $f(\log x) = s(x)$ we have

$$\begin{aligned} T_\gamma(\eta) &= s_{[\tau]} + o(1) \\ T_\beta(y) &= \frac{\alpha}{u^\alpha}\int_0^u s(x)(u-x)^{\alpha-1}dx \equiv C^{(\alpha)}(u) \end{aligned} \tag{2.14}$$

$C^{(\alpha)}(u)$ being the Cèsaró-transform of order α of the function $s(x)$.

Now by Theorem 1

$$\overline{\lim_{t\to\infty}} \; |s_{[\tau]} - C^{(\alpha)}(u)| \leqq L \cdot A_q^{(\alpha)}$$

where $\tau \to +\infty$, $u \to +\infty$, $\tau u^{-1} \to q > 0$, as $t \to +\infty$, and

$$A_q^{(\alpha)} = \begin{cases} \displaystyle\int_0^{-\log} (1-e^{-x})^\alpha dx + \int_{-\log q}^{+\infty} (1-(1-e^{-x})^\alpha)dx & \text{if } q < 1 \\ \displaystyle\log q + \int_0^\infty (1-(1-e^{-x})^\alpha)dx & \text{if } q \geqq 1 \end{cases}$$

This is a result analogous to a theorem of V. Garten [8] and A. Jakimowski [9].

EXAMPLE (iii). The $[J, f(x)]$-transformations were defined in [6] as follows: Let $\phi(x)$ be a function of bounded variation in $[0,1]$, $\phi(0+) = \phi(0) = 0$, $\phi(1-) = \phi(1) = 1$. The $[J, f(x)]$—transforms of a sequence $\{s_n\}$ $(n \geqq 1)$ $(s_n = a_1 + \dots + a_n)$ is

$$F_\phi(x) = \sum_{k=1}^{\infty} s_k \frac{x^k}{k!}\int_0^1 u^x \left(\log\frac{1}{u}\right)^k d\phi(u) \tag{2.15}$$

If we denote $x = e^y$, $u = e^{-e^{-v}}$, $\phi(u) = \beta(v)$, we have

$$F_\phi(x) = F_\phi(e^y) = \sum_{k=1}^{\infty} s_k \int_{-\infty}^{+\infty} \frac{e^{kv}}{k!} e^{-e^v} d(1-\beta(y-v)) \tag{2.16}$$

Now, if $ka_k = O(1)$, and $B(x) = \sum_{k=1}^{\infty} s_k(x^k/k!)e^{-x}$ is the usual Borel-transform of the sequence $\{s_k\}$, it is easy to show (see for example my paper [10]) that

$$\lim_{x \to +\infty} |B(x) - s_{[x]}| = 0 \tag{2.17}$$

thus

$$B(e^v) = O(v) \quad \text{as } v \to +\infty.$$

Therefore, if $\beta(v)$ satisfies the conditions of Theorem 1, it follows by (2.7) that the integral

$$\int_{-\infty}^{+\infty} B(e^v) d(1 - \beta(y - v))$$

is absolutely convergent, and thus we may write (2.16) in the equivalent form

$$F_\phi(x) = \int_{-\infty}^{+\infty} B(e^v) d(1 - \beta(y - v)). \tag{2.18}$$

Also, by (2.12) and (2.17),

$$\lim_{v \to +\infty} |B(e^v) - f(v)| = 0;$$

therefore, it is easily seen that

$$\lim_{y \to +\infty} |F_\phi(x) - T_\beta(y)| = 0, \tag{2.19}$$

where

$$T_\beta(y) = \int_{-\infty}^{+\infty} f(v) d(1 - \beta(y - v)).$$

Now, if $\gamma(x)$ is as in (2.11), $e^\eta = \tau$, and $L = \overline{\lim}_{k \to +\infty} |ka_k|$, by Theorem 1 and (2.19)

$$\overline{\lim_{t \to +\infty}} |s_{[\tau]} - F_\phi(x)| \leqq L \cdot A(\phi, q) \tag{2.20}$$

where $\tau \to +\infty$, $x \to +\infty$, $\tau x^{-1} \to q > 0$ as $t \to +\infty$, and

$$A(\phi, q) = \int_{-\infty}^{-\log q} |\phi(e^{-e^{-v}})| dv + \int_{-\log q}^{+\infty} |1 - \phi(e^{-e^{-v}})| dv$$

This is a more general result than Jakimowski's in [8], namely we do not suppose here that $\phi(x)$ is a monotonic function of x in $[0, 1]$, but only that the conditions of Theorem 1 are satisfied concerning $\beta(x) = \phi(e^{-e^{-x}})$. The monotonicity of $\phi(x)$ clearly implies our assumptions.

Example. (iv). Let $\beta(x)$ be as in (2.13) and $\gamma(x) = e^{-e^{-x}}$. Then if $e^y = u$, $e^{-e^{-\eta}} = \rho$, we have

$$T_\beta(y) = C^{(\alpha)}(u) \qquad \text{as in (2.14)}$$
$$T_\gamma(\eta) = A(\rho) \qquad \text{as in example (i)}$$

Now, by Theorem 1

$$\overline{\lim_{t\to\infty}}\ |C^{(\alpha)}(u) - A(\rho)| \leqq L \cdot A(q,\alpha)$$

where $u \to +\infty$, $\rho \to 1$, $u(1-\rho) \to q > 0$ if $t \to +\infty$,

$$A(q,\alpha) = \int_{-\infty}^{0} e^{-qe^{-x}}\,dx + \int_{0}^{\infty} |(1-e^{-x})^{\alpha} - e^{-qe^{-x}}|\,dx.$$

If $q \leqq \alpha$ the sign of absolute value in the second integral can be removed, since $e^{-u} \geqq 1 - u$ for all $u \geqq 0$. In particular, if $\alpha = q = 1$,

$$\overline{\lim_{t\to\infty}}\ |C^{(1)}(u) - A(\rho)| \leqq L \cdot A_1$$

where $u \to +\infty$, $\rho \to 1$, $u(1-\rho) \to 1$ as $t \to +\infty$, and an easy calculation yields

$$A_1 = 1 - C,$$

C being Euler's constant.

Concerning transforms $T_\beta(y)$ with a monotonic increasing function $\beta(x)$ we shall be able to prove the much more general result:

THEOREM 2. *Let $\beta(x)$ be an increasing function satisfying the conditions of Theorem 1 and let $s(x) \equiv f(\log x)$ be a real, or complex valued slowly oscillating function for $x > 0$ (i.e $s(x) - s(y) \to 0$, if $x \to +\infty$, $xy^{-1} \to 1$). Then for every fixed q, $-\infty < q < +\infty$,*

$$\overline{\lim_{t\to\infty}}\ |f(\eta) - T_\beta(y)| \leqq L^* \cdot A_q^*, \tag{2.21}$$

where $y = y(t) \to +\infty$, $\eta = \eta(t) \to +\infty$, $y - \eta \to q$ as $t \to +\infty$,

$$L^* = \lim_{\delta \downarrow 0}\ \overline{\lim_{y \geqq x+\delta \to \infty}}\ \left|\frac{f(y)-f(x)}{y-x}\right| \tag{2.22}$$

and

$$A_q^* = \int_{-\infty}^{q} \beta(x)dx + \int_{q}^{+\infty} (1-\beta(x))dx. \tag{2.23}$$

Moreover, the constant A_q^* in (2.21) is the best possible in the same sense as in Theorem 1.

REMARK. It is well known, that if $f(x)$ satisfies conditions of Theorem 1, then $s(x) \equiv f(\log x)$ is slowly oscillating and L^* in (2.22) is $\leqq$ than L in (2.4). By (2.23), A_q^* has the same value as A_q by Theorem 1 for the special case $T_y(\eta) = f(\eta)$. Thus, for increasing $\beta(x)$ Theorem 2 is a more general result.

Proofs. In the proof we shall need the following modification of the lemma of Agnew [9].

LEMMA. *Let $H(t,x)$ be a real function of the variables*

$$t,x(t>0;\ -\infty<x<+\infty)$$

satisfying the following conditions

$$\int_{-\infty}^{+\infty}|H(t,x)|\,dx \text{ exists for } t>0,$$

$$\lim_{t\to\infty}\int_{-\infty}^{c}|H(t,x)|\,dx=0 \qquad \text{for every } c,$$

and suppose

$$\overline{\lim_{t\to+\infty}}\int_{-\infty}^{+\infty}|H(t,x)|\,dx=A<+\infty.$$

Let $g(x)$ be any function of the real variable x, $(-\infty<x<+\infty)$ satisfying $g(x)=O(1)$ for $-\infty<x<+\infty$, and suppose

$$\overline{\lim_{x\to+\infty}}|g(x)|=L<+\infty.$$

Then for

$$T(t)=\int_{-\infty}^{+\infty}H(t,x)g(x)dx$$

we have

$$\overline{\lim_{t\to+\infty}}|T(t)|\leqq L\cdot A, \tag{2.24}$$

and the constant A is the best possible in the sense that there exists a real function $g(x)$ with $0<L<+\infty$ such that in (2.24) both sides are equal.

The proof of the lemma is the same as the very similar lemma of Rajagopal [10].

Proof of Theorem 1. By (1.1) we have

$$T_\beta(y)-T_\gamma(\eta)=\int_{-\infty}^{+\infty}f(x)d\{\gamma(\eta-x)-\beta(y-x)\};$$

by (2.3)

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{x}f'(u)du\,d\{\gamma(\eta-x)-\beta(y-x)\}.$$

By (2.2), (2.6) and (2.7) we may interchange the order of integrations and by (2.5) we obtain

$$=\int_{-\infty}^{+\infty}f'(u)\{\beta(y-u)-\gamma(\eta-u)\}du$$

$$\equiv\int_{-\infty}^{+\infty}f'(u)H(t,u)du,$$

with $y = y(t)$, $\eta = \eta(t)$. Now it is easy to check that $H(t,u)$ satisfies the conditions of the Lemma; thus we have

$$\overline{\lim_{t\to\infty}} \, |T_\beta(y) - T_\gamma(\eta)| \leqq L \cdot A_q,$$

where

$$A_q = \overline{\lim_{t\to\infty}} \int_{-\infty}^{+\infty} |\beta(y-u) - \gamma(\eta-u)|\, du,$$

and by (2.8) and Lebesgue's well known theorem

$$= \int_{-\infty}^{+\infty} |\beta(x) - \gamma(x-q)|\, dx.$$

Q.E.D.

Proof of Theorem 2. First, by a slight modification of a theorem of *R.* Schmidt [11], for all x, y satisfying $|x-y| \geqq \delta > 0$

$$|f(x) - f(y)| \leqq K_\delta \cdot |x-y|, \tag{2.25}$$

K being dependent only on δ. Thus the lim defining L^* in (2.22) certainly exists.

In order to prove our Theorem it is enough to show that (2.21) holds; the fact that A_q^* is the best possible constant satisfying (2.21) follows from the remark after Theorem 2. Let now be $\varepsilon > 0$ given; define $\delta > 0$, $x_0 > 0$ such, that for $x \geqq x_0$, $y \geqq x_0$, $|x-y| \geqq \delta$ (by (2.22))

$$|f(x) - f(y)| < (L^* + \varepsilon)|x-y|, \tag{2.26}$$

and for $x \geqq x_0$, $y \geqq x_0$ and $|x-y| \leqq \delta$ (since $f(\log x)$ is slowly oscillating)

$$|f(x) - f(y)| < \varepsilon. \tag{2.27}$$

Now

$$\begin{aligned} f(\eta) - T_\beta(y) &= \int_{-\infty}^{+\infty} (f(\eta) - f(x))\, d(1 - \beta(y-x)) \\ &= \int_{-\infty}^{x_0} + \int_{x_0}^{\eta-\delta} + \int_{\eta-\delta}^{\eta+\delta} + \int_{\eta+\delta}^{+\infty} = I_1 + I_2 + I_3 + I_4 \end{aligned} \tag{2.28}$$

Let $y, \eta > x_0 + \delta$.

By (2.25)

$$\begin{aligned} I_1 &= O\left(\int_{-\infty}^{x_0} (\eta - x)\, d(1 - \beta(y-x)) \right) \\ &= O\left\{ (\eta - y)(1 - \beta(y - x_0)) + \int_{y-x_0}^{+\infty} u\, d\beta(u) \right\} \end{aligned}$$

and by (2.5) and (2.6) we obtain easily

$$I_1 = o(1) \quad \text{as } y \to \infty. \tag{2.29}$$

By (2.26)

$$|I_2| \leqq (L^* + \varepsilon)\int_{x_0}^{\eta-\delta} (\eta - x)d(1 - \beta(y - x)) \leqq (L^* + \varepsilon)\int_{y-\eta}^{\infty} d\beta(u)\int_{y-\eta}^{u} dt$$

(2.30)

$$= (L^* + \varepsilon)\cdot\int_{y-\eta}^{\infty} (1 - \beta(u))du$$

In the same way

(2.31) $$|I_4| \leqq \left(\int_{-\infty}^{y-\eta} \beta(u)du\right)\cdot(L^* + \varepsilon).$$

By (2.27)

(2.32) $$|I_3| \leqq \varepsilon\int_{y-\eta-\delta}^{y-\eta+\delta} d\beta(u) \leqq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have by (2.28) – (2.32), as $t \to \infty, y \to \infty, y - \eta \to q$,

$$\overline{\lim_{t\to\infty}}\; |f(\eta) - T_\beta(y)| \leqq L^* \cdot A_q$$

Q.E.D.

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